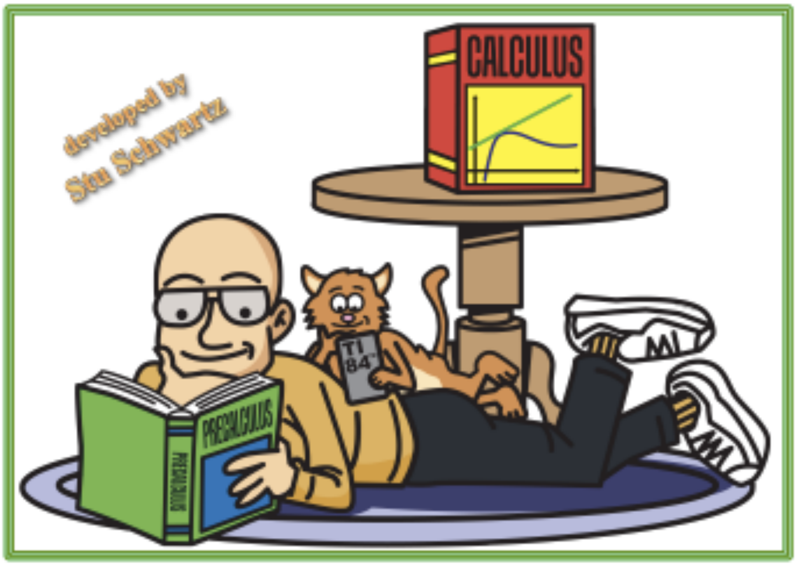
Name: ______________________________

# RU Ready for Some Calculus?

### A Precalculus Review




# AP Calculus Summer Prep Packet

Welcome to Calculus! This packet is for all students entering
AP Calculus AB or BC

Attached, you will find the basic learning targets from Algebra and Pre-Calculus that you are expected to remember BEFORE you come to class in the fall. For each topic addressed, this packet contains review examples, properties, definitions, and outline video tutorial links followed by practice problems. This material must be mastered in order for you to be successful in Calculus. You will be assessed at the beginning of the school. Year. Since this material is designed as a review, you are responsible for completing this packet on your own. The packet will be graded to assess the students' efforts to recall this information. Be sure to **show all your work!!**

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**A. Functions**

The lifeblood of precalculus is functions. A **function** is a set of points $(x, y)$ such that for every *x*, there is one and only one *y*. In short, in a function, the *x*-values cannot repeat while the *y*-values can. In AB Calculus, all of your graphs will come from functions.

The notation for functions is either "$y =$" or "$f(x) =$". In the $f(x)$ notation, we are stating a rule to find *y* given a value of *x*.

1. If $f(x) = x^2 - 5x + 8$, find a) $f(-6)$ b) $f\left(\frac{3}{2}\right)$ c) $\frac{f(x+h) - f(x)}{h}$

a) $f(-6) = (-6)^2 - 5(-6) + 8$
$36 + 30 + 8$
$74$

b) $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 8$
$\frac{9}{4} - \frac{15}{2} + 8$
$\frac{11}{4}$

c) $\frac{f(x+h) - f(x)}{h}$
$\frac{(x+h)^2 - 5(x+h) + 8 - (x^2 - 5x + 8)}{h}$
$\frac{x^2 + 2xh + h^2 - 5x - 5h + 8 - x^2 + 5x - 8}{5}$
$\frac{h^2 + 2xh - 5h}{h} = \frac{h(h + 2x - 5)}{h} = h + 2x - 5$

Need more? Watch this video for some "live" examples:

Functions do not always use the variable *x*. In calculus, other variables are used liberally.

2. If $A(r) = \pi r^2$, find a) $A(3)$ b) $A(2s)$ c) $A(r+1) - A(r)$

a) $A(3) = 9\pi$

b) $A(2s) = \pi(2s)^2 = 4\pi s^2$

c) $A(r+1) - A(r) = \pi(r+1)^2 - \pi r^2$
$\pi(2r+1)$

One concept that comes up in AP calculus is **composition of functions**. The format of a composition of functions is: plug a value into one function, determine an answer, and plug that answer into a second function.

3. If $f(x)=x^2-x+1$ and $g(x)=2x-1$, a) find $f(g(-1))$ b) find $g(f(-1))$ c) show that $f(g(x))\neq g(f(x))$

$g(-1)=2(-1)-1=-3$
$f(-3)=9+3+1=13$

$f(-1)=1+1+1=3$
$g(3)=2(3)-1=5$

$f(g(x))=f(2x-1)=(2x-1)^2-(2x-1)+1$
$=4x^2-4x+1-2x+1+1=4x^2-6x+3$
$g(f(x))=g(x^2-x+1)=2(x^2-x+1)-1$
$=2x^2-2x+1$

Need More???

Finally, expect to use **piecewise functions**. A piecewise function gives different rules, based on the value of $x$.

4. If $f(x)=\begin{cases} x^2-3, & x\geq 0 \\ 2x+1, & x<0 \end{cases}$, find a) $f(5)$ b) $f(2)-f(-1)$ c) $f(f(1))$

$f(5)=25-3=22$

$f(2)-f(-1)=1-(-1)=2$

$f(1)=-2,\ f(-2)=-3$

Need more help???

## A. Function Assignment

Given $f(x) = 4x - x^2$ , find

1. $f(2) =$

2. $f(4) - f(-4) =$

3. $\sqrt{f\left(\frac{3}{2}\right)} =$

4. $f(t) =$

5. $f(x+3) =$

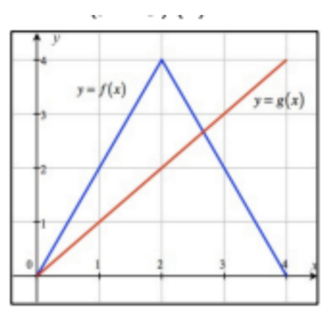
6. $\frac{f(x+h)-f(x)}{h} =$

Given $V(r) = \frac{4}{3}\pi r^3$, find

7. $V(2) =$

8. $V\left(\frac{3}{4}\right) =$

Given $f(x)$ $and$ $g(x)$ are given in the graph to the right, find:




9. $g(2) =$

10. $f - g(3) =$

11. $f(g(3)) =$

12. If $f(x) = x^2 - 5x + 3$ $and$ $g(x) = 1 - 2x$, then $f(g(x)) =$

Given: If $f(x) = \begin{cases} \sqrt{x+2} - 2, & x \geq 2 \\ x^2 - 1, & 0 \leq x < 2 \\ -x, & x < 0 \end{cases}$, find

13. $f(0) - f(2) =$

14. $\sqrt{5 - f(-4)} =$

15. $f(f(3)) =$

## B. Domain and Range

First, since questions in calculus usually ask about behavior of functions in intervals, understand that intervals can be written with a description in terms of $<, \leq, >, \geq$ or by using **interval notation**.

| Description | Interval notation | Description | Interval notation | Description | Interval notation |
|---|---|---|---|---|---|
| $x > a$ | $(a, \infty)$ | $x \leq a$ | $(-\infty, a]$ | $a \leq x < b$ | $[a, b)$ |
| $x \geq a$ | $[a, \infty)$ | $a < x < b$ | $(a, b)$ - open interval | $a < x \leq b$ | $(a, b]$ |
| $x < a$ | $(-\infty, a)$ | $a \leq x \leq b$ | $[a, b]$ - closed interval | All real numbers | $(-\infty, \infty)$ |

If a solution is in one interval or the other, interval notation will use the connector $\cup$. So $x \leq 2$ or $x > 6$ would be written $(-\infty, 2] \cup (6, \infty)$ in interval notation. Solutions in intervals are usually written in the easiest way to define it. For instance, saying that $x < 0$ or $x > 0$ or $(-\infty, 0) \cup (0, \infty)$ is best expressed as $x \neq 0$.

More Help?? (Interval Notation)

https://www.youtube.com/watch?v=Ww7xtG2S7IM

The **domain of a function** is the set of allowable $x$-values. The domain of a function $f$ is $(-\infty, \infty)$ except for values of $x$ which create a zero in the denominator, an even root of a negative number or a logarithm of a non-positive number. The domain of $a^{p(x)}$ where $a$ is a positive constant and $p(x)$ is a polynomial is $(-\infty, \infty)$.

• Find the domain of the following functions using interval notation:

1. $f(x) = x^2 - 4x + 4$

   $\boxed{(-\infty, \infty)}$

2. $y = \dfrac{6}{x - 6}$

   $\boxed{x \neq 6}$

3. $y = \dfrac{2x}{x^2 - 2x - 3}$

   $\boxed{x \neq -1, x \neq 3}$

4. $y = \sqrt{x + 5}$

   $\boxed{[-5, \infty)}$

5. $y = \sqrt[3]{x + 5}$

   $\boxed{(-\infty, \infty)}$

6. $y = \dfrac{x^2 + 4x + 6}{\sqrt{2x + 4}}$

   $\boxed{(-2, \infty)}$

Need some help??? (Domain and Range From a Graph)

https://www.youtube.com/watch?v=fyROLkZc75E

The **range of a function** is the set of allowable *y*-values. Finding the range of functions algebraically isn't as easy (it really is a calculus problem), but visually, it is the [lowest possible *y*-value, highest possible *y*-value]. Finding the range of some functions are fairly simple to find if you realize that the range of $y = x^2$ is $[0,\infty)$ as any positive number squared is positive. Also the range of $y = \sqrt{x}$ is also positive as the domain is $[0,\infty)$ and the square root of any positive number is positive. The range of $y = a^x$ where *a* is a positive constant is $(0,\infty)$ as constants to powers must be positive.

• Find the range of the following functions using interval notation:

7. $y = 1 - x^2$

$(-\infty,1]$

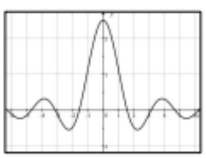
8. $y = \dfrac{1}{x^2}$

$(0,\infty)$

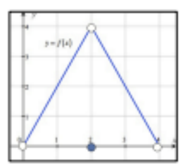
9. $y = \sqrt{x-8} + 2$

$[2,\infty)$

• Find the domain and range of the following functions using interval notation.

10. 

Domain: $(-\infty,\infty)$
Range: $[-0.5,2.5]$

11. 

Domain: $(0,4)$
Range: $[0,4)$

Need some serious help?? (Domain and Range of Rational Functions)
https://www.youtube.com/watch?v=Veq5BBnfMPQ

## B. Domain and Range Assignment

Find the domain of the following functions using interval notation:

16. $f(x) = 3$

17. $f(x) = x^3 - x^2 + x$

18. $f(x) = \frac{x^3 - x^2 + x}{x}$

19. $f(x) = \frac{x-4}{x^2-16}$

20. $y = \sqrt{x+3}$

21. $y = 3^x$

22. $y = log\ (x - 2)$

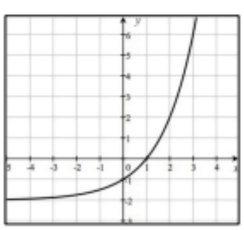
Find the range of the following functions.

23. $f(x) = x^4 + x^2 - 1$

24. $y = 2^x$

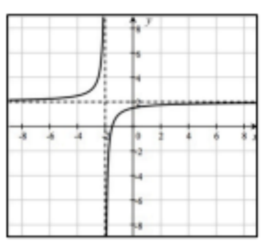
Find the domain and range of the following functions using interval notation.

25.

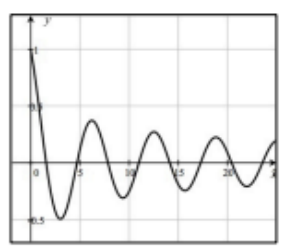


y = -2

26.



27.



## C. Even and Odd Functions

**Functions that are even** have the characteristic that for all $a$, $f(-a)=f(a)$. What this says is that plugging in a positive number $a$ into the function or a negative number $-a$ into the function makes no difference … you will get the same result. Even functions are symmetric to the $y$-axis.

**Functions that are odd** have the characteristic that for all $a$, $f(-a)=-f(a)$. What this says is that plugging in a negative number $-a$ into the function will give you the same result as plugging in the positive number and taking the negative of that. So, odd functions are symmetric to the origin. If a graph is symmetric to the $x$-axis, it is not a function because it fails the vertical-line test.

1. Of the common functions in section 3, which are even, which are odd, and which are neither?

> Even: $y=a$, $y=x^2$, $y=|x|$, $y=\cos x$ Odd: $y=x$, $y=x^3$, $y=\frac{1}{x}$, $y=\sin x$
>
> Neither: $y=\sqrt{x}$, $y=\ln x$, $y=e^x$, $y=e^{-x}$

2. Show that the following functions are even:

a) $f(x)=x^4-x^2+1$

> $f(-x)=(-x)^4-(-x)^2+1$
> $=x^4-x^2+1=f(x)$

b) $f(x)=\left|\frac{1}{x}\right|$

> $f(-x)=\left|\frac{1}{-x}\right|=\left|\frac{1}{x}\right|=f(x)$

c) $f(x)=x^{2/3}$

> $f(-x)=(-x)^{2/3}=\left(\sqrt[3]{-x}\right)^2$
> $=\left(\sqrt[3]{x}\right)^2=f(x)$

3. Show that the following functions are odd:

a) $f(x)=x^3-x$

> $f(-x)=(-x)^3+x$
> $=x-x^3=-f(x)$

b) $f(x)=\sqrt[3]{x}$

> $f(-x)=\sqrt[3]{-x}=-\sqrt[3]{x}=-f(x)$

c) $f(x)=e^x-e^{-x}$

> $f(-x)=e^{-x}-e^x=-\left(e^x-e^{-x}\right)=-f(x)$

4. Determine if $f(x)=x^3-x^2+x-1$ is even, odd, or neither. Justify your answer.

> $f(-x)=-x^3-x^2-x-1\neq f(x)$ so $f$ is not even. $-f(x)=-x^3+x^2-x-1\neq f(-x)$ so $f$ is not odd.

Graphs may not be functions and yet have $x$-axis or $y$-axis or both. Equations for these graphs are usually expressed in "implicit form" where it is not expressed as "$y=$" or "$f(x)=$". If the equation does not change after making the following replacements, the graph has these symmetries:

$x$-axis: $y$ with $-y$  $y$-axis: $x$ with $-x$  origin: both $x$ with $-x$ and $y$ with - $y$

5. Determine the symmetry for $x^2+xy+y^2=0$.

> $x-$axis: $x^2+x(-y)+(-y)^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $x-$axis
> $y-$axis: $(-x)^2+(-x)(y)+y^2=0\Rightarrow x^2-xy+y^2=0$ so not symmetric to $y-$axis
> origin: $(-x)^2+(-x)(-y)+y^2=0\Rightarrow x^2+xy+y^2=0$ so symmetric to origin

Need more?? https://www.youtube.com/watch?v=fKyBOLsqRlo

## C. Even and Odd Functions – Assignment

Show work to determine if the following functions are even, odd, or neither.

28. $f(x) = 7$

29. $f(x) = 2x^2 - 4x$

30. $f(x) = -3x^3 - 2x$

31. $f(x) = \sqrt{x + 1}$

32. $f(x) = \sqrt{x^2 + 1}$

33. $f(x) = |8x|$

## D. Special Factorization

While factoring skills were more important in the days when A topics were specifically tested, students still must know how to factor. The special forms that occur most regularly are:

| |
|---|
| **Common factor**: $x^3+x^2+x=x\left(x^2+x+1\right)$ |
| **Difference of squares**: $x^2-y^2=(x+y)(x-y)$ or $x^{2n}-y^{2n}=\left(x^n+y^n\right)\left(x^n-y^n\right)$ |
| **Perfect squares**: $x^2+2xy+y^2=(x+y)^2$ |
| **Perfect squares**: $x^2-2xy+y^2=(x-y)^2$ |
| **Sum of cubes**: $x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ - Trinomial unfactorable |
| **Difference of cubes**: $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$ - Trinomial unfactorable |
| **Grouping**: $xy+xb+ay+ab=x(y+b)+a(y+b)=(x+a)(y+b)$ |

The term "factoring" usually means that coefficients are rational numbers. For instance, $x^2-2$ could technically be factored as $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$ but since $\sqrt{2}$ is not rational, we say that $x^2-2$ is not factorable. It is important to know that $x^2+y^2$ is unfactorable.

• Completely factor the following expressions.

1. $4a^2+2a$

   $\boxed{2a(2a+1)}$

2. $x^2+16x+64$

   $\boxed{(x+8)^2}$

3. $4x^2-64$

   $\boxed{4(x+4)(x-4)}$

4. $5x^4-5y^4$

   $\boxed{5\left(x^2+y^2\right)(x+y)(x-y)}$

5. $16x^2-8x+1$

   $\boxed{(4x-1)^2}$

6. $9a^4-a^2b^2$

   $\boxed{a^2(3a+b)(3a-b)}$

7. $2x^2-40x+200$

   $\boxed{2(x-10)^2}$

8. $x^3-8$

   $\boxed{(x-2)\left(x^2+2x+4\right)}$

9. $8x^3+27y^3$

   $\boxed{(2x+3y)\left(4x^2-6xy+9y^2\right)}$

10. $x^4-11x^2-80$

    $\boxed{(x+4)(x-4)\left(x^2+5\right)}$

11. $x^4-10x^2+9$

    $\boxed{(x+1)(x-1)(x+3)(x-3)}$

12. $36x^2-64$

    $\boxed{4(3x+4)(3x-4)}$

13. $x^3-x^2+3x-3$

    $x^2(x-1)+3(x-1)$

    $(x-1)\left(x^2+3\right)$

14. $x^3+5x^2-4x-20$

    $x^2(x+5)-4(x+5)$

    $(x+5)(x-2)(x+2)$

15. $9-\left(x^2+2xy+y^2\right)$

    $9-(x+y)^2$

    $(3+x+y)(3-x-y)$

More help??

https://www.youtube.com/watch?v=KUMhpKGwpCY

### D. Special Factorization – Assignment

Completely factor the following expressions.

34. $x^2 - 5x - 24$

35. $x^2 - 81$

36. $x^3 - 25x$

37. $30x - 9x^2 - 25$

38. $33x^8 - 3$

39. $16x^4 - 24x^2y + 9y^2$

40. $9a^4 - a^2b^2$

41. $4x^4 + 7x^2 - 36$

42. $8x^3 - 27$

43. $x^3 - xy^2 + x^2y - y^3$

44. $x^5 + x^3 + x^2 + 1$

45. $x^6 - 1$

***For AP Calculus students ONLY.***

Factoring Fractional Exponents.

Video Examples

46. $3x^{1/2} + 6x^{3/2}$

47. $x^{5/2} - 4x^{1/2}$

48. $2(x-3)^{5/2} + 5x(x-3)^{1/2}$

49. $x^{-2} + 3x^{-4}$

50. $3(x+2)^{-1/2} + 4x(x+2)^{1/2}$

51. $x(2x-1)^{3/2} - 9(2x-1)^{-1/2}$

## E. Linear Functions

Probably the most important concept from precalculus that is required for differential calculus is that of linear functions. The formulas you need to know backwards and forwards are:

**Slope**: Given two points $(x_1, y_1)$ and $(x_2, y_2)$, the slope of the line passing through the points can be written as: $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

**Slope intercept form**: the equation of a line with slope $m$ and $y$-intercept $b$ is given by $y = mx + b$.

**Point-slope form**: the equation of a line passing through the points $(x_1, y_1)$ and slope $m$ is given by $y - y_1 = m(x - x_1)$. While you might have preferred the simplicity of the $y = mx + b$ form in your algebra course, the $y - y_1 = m(x - x_1)$ form is far more useful in calculus.

**Intercept form:** the equation of a line with $x$-intercept $a$ and $y$-intercept $b$ is given by $\frac{x}{a} + \frac{y}{b} = 1$.

**General form**: $Ax + By + C = 0$ where $A$, $B$ and $C$ are integers. While your algebra teacher might have required your changing the equation $y - 1 = 2(x - 5)$ to general form $2x - y - 9 = 0$, you will find that on the AP calculus test, it is sufficient to leave equations for a lines in point-slope form and it is recommended not to waste time changing it unless you are specifically told to do so.

**Parallel lines** Two distinct lines are parallel if they have the same slope: $m_1 = m_2$.

**Normal lines:** Two lines are normal (perpendicular) if their slopes are negative reciprocals: $m_1 \cdot m_2 = -1$.

**Horizontal lines** have slope zero. **Vertical lines** have no slope (slope is undefined).

1. Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

a. $m = -4, (1,2)$

$y - 2 = -4(x - 1) \Rightarrow y = -4x + 6$

b. $m = \frac{2}{3}, (5,1)$

$y - 1 = \frac{2}{3}(x - 5) \Rightarrow y = \frac{2x}{3} - \frac{7}{3}$

c. $m = 0, \left(-\frac{1}{2}, \frac{3}{4}\right)$

$y = -\frac{3}{4}$

2. Find the equation of the line in slope-intercept form, passing through the following points.

a. $(4,5)$ and $(-2,-1)$

$m = \frac{5+1}{4+2} = 1$

$y - 5 = x - 4 \Rightarrow y = x + 1$

b. $(0,-3)$ and $(-5,3)$

$m = \frac{3+3}{-5-0} = \frac{-6}{5}$

$y + 3 = \frac{-6}{5}x \Rightarrow y = \frac{-6}{5}x - 3$

c. $\left(\frac{3}{4}, -1\right)$ and $\left(1, \frac{1}{2}\right)$

$m = \left(\frac{\frac{1}{2}+1}{1-\frac{3}{4}}\right)\left(\frac{4}{4}\right) = \frac{2+4}{4-3} = 6$

$y - \frac{1}{2} = 6(x - 1) \Rightarrow y = 6x - \frac{11}{2}$

3. Write equations of the line through the given point a) parallel and b) normal to the given line.

a. $(4,7)$, $4x - 2y = 1$

$y = 2x - \frac{1}{2} \Rightarrow m = 2$

a) $y - 7 = 2(x - 4)$ b) $y - 7 = \frac{-1}{2}(x - 4)$

b. $\left(\frac{2}{3}, 1\right)$, $x + 5y = 2$

$y = \frac{-1}{5}x + 2 \Rightarrow m = \frac{-1}{5}$

a) $y - 1 = \frac{-1}{5}\left(x - \frac{2}{3}\right)$ b) $y - 1 = 5\left(x - \frac{2}{3}\right)$

More, more, more? Videos to help you with assignment ☺️

Equation of Line Given Two Points

https://www.youtube.com/watch?v=4vXqMsvPSv4

Equation of Line Given Point and Slope

https://www.youtube.com/watch?v=AqONrrPhJvk

Equation of a Line Through a Point Parallel or Perpendicular to Another Line

https://www.youtube.com/watch?v=TrONIeOpJHg

## E. Linear Functions – Assignment

Find the equation of the line in slope-intercept form, with the given slope, passing through the given point.

52. $m = -7$, $(-3, -7)$ 53. $m = -\frac{1}{2}, (2, -8)$ 54. $m = \frac{2}{3}, \left(-6, \frac{1}{3}\right)$

Find the equation of the line in slope-intercept form passing through the following points.

55. $(-3, 6)$ *and* $(-1, 2)$ 56. $\left(-2, \frac{2}{3}\right)$ *and* $\left(\frac{1}{2}, 1\right)$

Write the equations of the line through the given point a) parallel and b) perpendicular to the given line.

57. $(-6, 2)$; $5x + 2y = 7$ 58. $(-3, -4)$; $y = -2$

59. Find an equation of the line containing (4, -2) and parallel to the line containing (-1, 4) and (2, 3). Put your answer in point-slope form.

60. Find *k* if the lines $3x - 5y = 9$ *and* $2x + ky = 11$ are: a) parallel and b) perpendicular.

## F. Solving Quadratic Equations

Solving quadratics in the form of $ax^2+bx+c=0$ usually show up on the AP exam in the form of expressions that can easily be factored. But occasionally, you will be required to use the quadratic formula. When you have a quadratic equation, factor it, set each factor equal to zero and solve. If the quadratic equation doesn't factor or if factoring is too time-consuming, use the quadratic formula:

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. The **discriminant** $b^2-4ac$ will tell you how many solutions the quadratic has:

$$b^2-4ac\begin{cases}>0,\ 2\text{ real solutions (if a perfect square, the solutions are rational)}\\=0,\ 1\text{ real solution}\\<0,\ 0\text{ real solutions (or 2 imaginary solutions, but AP calculus does not deal with imaginaries)}\end{cases}$$

1. Solve for $x$.

a. $x^2+3x+2=0$

$(x+2)(x+1)=0$
$x=-2, x=-1$

b. $x^2-10x+25=0$

$(x-5)^2=0$
$x=5$

c. $x^2-64=0$

$(x-8)(x+8)=0$
$x=8, x=-8$

d. $2x^2+9x=18$

$(2x-3)(x+6)=0$
$x=\frac{3}{2}, x=-6$

e. $12x^2+23x=-10$

$(4x+5)(3x+2)=0$
$x=-\frac{5}{4}, x=-\frac{2}{3}$

f. $48x-64x^2=9$

$(8x-3)^2=0$
$x=\frac{3}{8}$

g. $x^2+5x=2$

$x=\dfrac{-5\pm\sqrt{25+8}}{2}$
$x=\dfrac{-5\pm\sqrt{33}}{2}$

h. $8x-3x^2=2$

$x=\dfrac{8\pm\sqrt{64-24}}{6}$
$x=\dfrac{8\pm2\sqrt{10}}{6}=\dfrac{4\pm\sqrt{10}}{3}$

i. $6x^2+5x+3=0$

$x=\dfrac{-5\pm\sqrt{25-72}}{12}=\dfrac{-5\pm\sqrt{-47}}{12}$
No real solutions

j. $x^3-3x^2+3x-9=0$

$x^2(x-3)-3(x-3)=0$
$(x-3)(x^2-3)=0$
$x=3,\ x=\pm\sqrt{3}$

k. $\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}$

$6x\left(\dfrac{x}{3}-\dfrac{5}{2}=\dfrac{-3}{x}\right)$
$2x^2-15x+18=0$
$(2x-3)(x-6)=0$
$x=\frac{3}{2},\ x=6$

l. $x^4-7x^2-8=0$

$(x^2-8)(x^2+1)=0$
$x=\pm\sqrt{8}=\pm2\sqrt{2}$

**Need More???**

https://www.youtube.com/watch?v=glQPvmgN5Tw&t=3s

https://www.youtube.com/watch?v=5Nf_2rYbI-w&t=2s

## F. Solving Quadratic Equations – Assignment

Solve for x.

61. $x^2 + 7x - 18 = 0$

62. $2x^2 - 72 = 0$

63. $12x^2 - 5x = 2$

64. $81x^2 + 72x + 16 = 0$

65. $x^2 + 10x = 7$

66. $3x - 4x^2 = -5$

67. $x^3 - 5x^2 + 5x - 25 = 0$

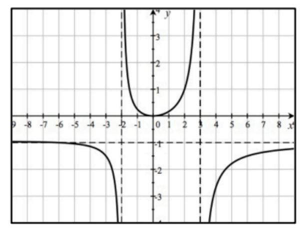
68. $2x^4 - 15x^3 + 18x^2 = 0$

## G. Asymptotes and Holes

Rational functions in the form of $y=\frac{p(x)}{q(x)}$ possibly have vertical asymptotes, lines that the graph of the curve approach but never cross. To find the **vertical asymptotes**, factor out any common factors of numerator and denominator, reduce if possible, and then set the denominator equal to zero and solve.

**Horizontal asymptotes** are lines that the graph of the function approaches when $x$ gets very large or very small. While you learn how to find these in calculus, a rule of thumb is that if the highest power of $x$ is in the denominator, the horizontal asymptote is the line $y = 0$. If the highest power of $x$ is both in numerator and denominator, the horizontal asymptote will be the line $y=\frac{\text{highest degree coefficient in numerator}}{\text{highest degree coefficient in denominator}}$. If the highest power of $x$ is in the numerator, there is no horizontal asymptote, but a slant asymptote which is not used in calculus.

1) Find any vertical and horizontal asymptotes for the graph of $y=\frac{-x^2}{x^2-x-6}$.

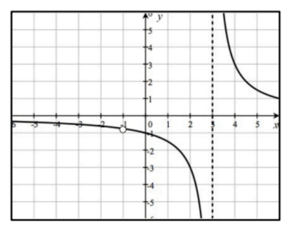


$$y=\frac{-x^2}{x^2-x-6}=\frac{-x^2}{(x-3)(x+2)}$$

Vertical asymptotes: $x-3=0 \Rightarrow x=3$ and $x+2=0 \Rightarrow x=-2$
Horizontal asymptotes: Since the highest power of $x$ is 2 in both numerator and denominator, there is a horizontal asymptote at $y=-1$.

This is confirmed by the graph to the right. Note that the curve actually crosses its horizontal asymptote on the left side of the graph.

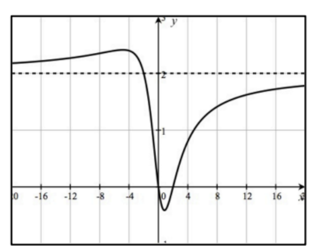
2) Find any vertical and horizontal asymptotes for the graph of $y=\frac{3x+3}{x^2-2x-3}$.



$$y=\frac{3x+3}{x^2-2x-3}=\frac{3(x+1)}{(x-3)(x+1)}=\frac{3}{x-3}$$

Vertical asymptotes: $x-3=0 \Rightarrow x=3$. Note that since the $(x+1)$ cancels, there is no vertical asymptote at $x = 1$, but a hole (sometimes called a removable discontinuity) in the graph.
Horizontal asymptotes: Since there the highest power of $x$ is in the denominator, there is a horizontal asymptote at $y = 0$ (the $x$-axis). This is confirmed by the graph to the right.

3) Find any vertical and horizontal asymptotes for the graph of $y=\frac{2x^2-4x}{x^2+4}$.



$$y=\frac{2x^2-4x}{x^2+4}=\frac{2x(x-2)}{x^2+4}$$

Vertical asymptotes: None. The denominator doesn't factor and setting it equal to zero has no solutions.
Horizontal asymptotes: Since the highest power of $x$ is 2 in both numerator and denominator, there is a horizontal asymptote at $y = 2$. This is confirmed by the graph to the right.

**** A HOLE is also known as REMOVABLE discontinuity.**
**** A VERTICAL ASYMPTOTE is also known as NON-REMOVABLE discontinuity.**

**Need more???**

https://www.educreations.com/lesson/view/pc-notes-2-6-day-1/60375266/?s=eoohta

https://www.educreations.com/lesson/view/pc-notes-2-6-day-2/60376716/?s=0qputx

## G. Asymptotes – Assignment

Find any holes first. Then locate all vertical, horizontal, and slant asymptotes. (Label the holes and Vertical Asymptotes as REMOVABLE OR NON-REMOVABLE discontinuity).

69. $y = \frac{x-1}{x+5}$

70. $y = \frac{2x+16}{x+8}$

71. $y = \frac{2x^2+6x}{x^2+5x+6}$

72. $y = \frac{x}{x^2-25}$

73. $y = \frac{x^3}{x^2+4}$

74. $y = \frac{2x^2-x+3}{x+1}$

## H. Simplifying Complex Fractions

Calculus frequently uses **complex fractions**, which are fractions within fractions. Answers are never left with complex fractions and they must be eliminated. There are two methods to eliminate complex fractions:

When the problem is in the form of $\frac{\frac{a}{b}}{\frac{c}{d}}$, we can "flip the denominator" and write it as $\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$.

However, this does not work when the numerator and denominator are not single fractions. The best way is to eliminate the complex fractions in all cases is to find the LCD (lowest common denominator) of all the fractions in the complex fraction. Multiply all terms by this LCD and you are left with a fraction that is magically no longer complex. **Important:** Note that $\frac{x^{-1}}{y^{-1}}$ can be written as $\frac{y}{x}$ but $\frac{1+x^{-1}}{y^{-1}}$ must be written as $\frac{1+\frac{1}{x}}{\frac{1}{y}}$.

• Eliminate the complex fractions.

1. $\dfrac{\frac{2}{3}}{\frac{5}{6}}$

$$\left(\frac{\frac{2}{3}}{\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{4}{5}$$

2. $\dfrac{1+\frac{2}{3}}{1+\frac{5}{6}}$

$$\left(\frac{1+\frac{2}{3}}{1+\frac{5}{6}}\right)\left(\frac{6}{6}\right)=\frac{6+4}{6+5}=\frac{10}{11}$$

3. $\dfrac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}$

$$\left(\frac{\frac{3}{4}+\frac{5}{3}}{2-\frac{1}{6}}\right)\left(\frac{12}{12}\right)=\frac{9+20}{24-2}=\frac{29}{22}$$

4. $\dfrac{1+\frac{1}{2}x^{-1}}{1+\frac{1}{3}x^{-1}}$

$$\left(\frac{1+\frac{1}{2x}}{1+\frac{1}{3x}}\right)\left(\frac{6x}{6x}\right)=\frac{6x+3}{6x+2}$$

5. $\dfrac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}$

$$\left(\frac{x-\frac{1}{2x}}{x^2+\frac{1}{4x^2}}\right)\left(\frac{4x^2}{4x^2}\right)=\frac{4x^3-2x}{4x^4+1}$$

6. $\dfrac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}$

$$\left(\frac{\frac{2}{5}x^{5/3}}{\frac{5}{3}}\right)\left(\frac{15}{15}\right)=\frac{6x^{5/3}}{25}$$

7. $\dfrac{x^{-3}+x}{x^{-2}+1}$

$$\left(\frac{\frac{1}{x^3}+x}{\frac{1}{x^2}+1}\right)\left(\frac{x^3}{x^3}\right)=\frac{1+x^4}{x+x^3}$$

8. $\dfrac{\frac{1}{2}(2x+5)^{-2/3}}{\frac{-2}{3}}$

$$\left(\frac{\frac{1}{2}}{\frac{-2}{3}(2x+5)^{2/3}}\right)\frac{6}{6}=\frac{-3}{4(2x+5)^{2/3}}$$

9. $\dfrac{(x-1)^{1/2}-\frac{x(x-1)^{-1/2}}{2}}{x-1}$

$$\left(\frac{(x-1)^{1/2}-\frac{x}{2(x-1)^{1/2}}}{x-1}\right)\left[\frac{2(x-1)^{1/2}}{2(x-1)^{1/2}}\right]$$
$$\frac{2(x-1)-x}{2(x-1)^{3/2}}=\frac{x-2}{2(x-1)^{3/2}}$$

Videos ~

https://www.youtube.com/watch?v=PpSyx-brMyg

## H. Eliminating Complex Fractions - Assignment

Eliminate the complex fractions. Do no leave any negative exponents.

75. $\frac{\frac{5}{8}}{\frac{-2}{3}} =$

76. $\frac{4-\frac{2}{9}}{3+\frac{4}{3}} =$

77. $\frac{2+\frac{7}{2}+\frac{3}{5}}{5-\frac{3}{4}} =$

78. $\frac{x-\frac{1}{x}}{x+\frac{1}{x}} =$

79. $\frac{1+x^{-1}}{1-x^{-2}} =$

80. $\frac{x^{-1}+y^{-1}}{x+y} =$

81. $\frac{x^{-2}+x^{-1}+1}{x^{-2}-x} =$

82. $\frac{\frac{1}{3}(3x-4)^{-3/4}}{-\frac{3}{4}} =$

83. $\frac{2x(2x-1)^{1/2}-2x^2(2x-1)^{-1/2}}{(2x-1)} =$

## I. Adding Fractions and Solving Fractional Equations

There are two major problem types with fractions: Adding/subtracting fractions and solving fractional equations. Algebra has taught you that in order to add fractions, you need to find an LCD and *multiply each fraction by one* … in such a way that you obtain the LCD in each fraction. However, when you solve fractional equations (equations that involve fractions), you still find the LCD but you *multiply every term by the LCD*. When you do that, all the fractions disappear, leaving you with an equation that is hopefully solvable. Answers should be checked in the original equation.

1. a. Combine: $\frac{x}{3}-\frac{x}{4}$

LCD: 12 $\quad \frac{x}{3}\left(\frac{4}{4}\right)-\frac{x}{4}\left(\frac{3}{3}\right)$

$\frac{4x-3x}{12}=\frac{x}{12}$

b. Solve: $\frac{x}{3}-\frac{x}{4}=12$

$12\left(\frac{x}{3}\right)-12\left(\frac{x}{4}\right)=12(12)$

$4x-3x=144\Rightarrow x=144$

$x=144:\frac{144}{3}-\frac{144}{4}=48-36=12$

2. a. Combine $x+\frac{6}{x}$

LCD: $x$ $\quad x\left(\frac{x}{x}\right)+\frac{6}{x}$

$\frac{x^2+6}{x}$

b. Solve: $x+\frac{6}{x}=5$

$x(x)+x\left(\frac{6}{x}\right)=5x$

$x^2+6=5x\Rightarrow x^2-5x+6=0$

$(x-2)(x-3)=0\Rightarrow x=2,\ x=3$

$x=2:2+\frac{6}{2}=2+3=5 \quad x=3:3+\frac{6}{3}=3+2=5$

3. a. Combine: $\frac{12}{x+2}-\frac{4}{x}$

LCD: $x(x+2)$ $\quad \left(\frac{12}{x+2}\right)\left(\frac{x}{x}\right)-\frac{4}{x}\left(\frac{x+2}{x+2}\right)$

$\frac{12x-4x-8}{x(x+2)}$

$\frac{8x-8}{x(x+2)}$

b. Solve $\frac{12}{x+2}-\frac{4}{x}=1$

$\frac{12}{x+2}(x)(x+2)-\frac{4}{x}(x)(x+2)=1(x)(x+2)$

$12x-4x-8=x^2+2x\Rightarrow x^2-6x+8=0$

$(x-2)(x-4)=0\Rightarrow x=2,4$

$x=2:\frac{12}{4}-\frac{4}{2}=3-2=1 \quad x=4:\frac{12}{6}-\frac{4}{4}=2-1=1$

4. a. $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}$

LCD: $2(x-3)^2$

$\frac{x}{2(x-3)}\left(\frac{x-3}{x-3}\right)-\frac{3}{(x-3)^2}\left(\frac{2}{2}\right)$

$\frac{x^2-3x-6}{2(x-3)^2}$

b. Solve $\frac{x}{2x-6}-\frac{3}{x^2-6x+9}=\frac{x-2}{3x-9}$

$\left[\frac{x}{2(x-3)}-\frac{3}{(x-3)^2}=\frac{x-2}{3(x-3)}\right]6(x-3)^2$

$3x(x-3)-18=2(x-3)(x-2)$

$3x^2-9x-18=2x^2-10x+12$

$x^2+x-30=0\Rightarrow(x+6)(x-5)=0\Rightarrow x=-6,5$

$x=-6:\frac{-6}{-18}-\frac{3}{81}=\frac{-8}{-27} \quad x=5:\frac{5}{4}-\frac{3}{4}=\frac{3}{6}$

**Video:**

https://www.educreations.com/lesson/view/pc-notes-2-7/60375171/?s=6ig4zm

## I. Adding Fractions and Solving Fractional Equations – Assignments

84. a. Combine: $\frac{2}{3}-\frac{1}{x}$

b. Solve: $\frac{2}{3}-\frac{1}{x}=\frac{5}{6}$

85. a. Combine: $\frac{1}{x-3}+\frac{1}{x+3}$

b. Solve: $\frac{1}{x-3}+\frac{1}{x+3}=\frac{10}{x^2-9}$

86. a. Combine: $\frac{5}{2x}-\frac{5}{3x+15}$

b. Solve: $\frac{5}{2x}-\frac{5}{3(x+5)}=\frac{5}{x}$

87. a. Combine: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}$

b. Solve: $\frac{2x-1}{x-1}-\frac{3x}{2x+1}=\frac{x^2+11}{2x^2-x-1}$

## J. Exponential and Logarithmic Functions

Calculus spends a great deal of time on exponential functions in the form of $b^x$. Don't expect that when you start working with them in calculus, your teacher will review them. So learn them now! Students must know that the definition of a **logarithm** is based on exponential equations. If $y = b^x$ then $x = \log_b y$. So when you are trying to find the value of $\log_2 32$, state that $\log_2 32 = x$ and $2^x = 32$ and therefore $x = 5$.

If the base of a log statement is not specified, it is defined to be 10. When we asked for log 100, we are solving the equation: $10^x = 100$ and $x = 2$. The function $y = \log x$ has domain $(0,\infty)$ and range $(-\infty,\infty)$. In calculus, we primarily use logs with base *e*, which are called natural logs (ln). So finding ln 5 is the same as solving the equation $e^x = 5$. Students should know that the value of $e = 2.71828\ldots$

There are three rules that students must keep in mind that will simplify problems involving logs and natural logs. These rules work with logs of any base including natural logs.

i. $\log a + \log b = \log(a \cdot b)$ ii. $\log a - \log b = \log\left(\frac{a}{b}\right)$ iii. $\log a^b = b\log a$

1. Find a. $\log_4 8$

$\log_4 8 = x$
$4^x = 8 \Rightarrow 2^{2x} = 2^3$
$x = \frac{3}{2}$

b. $\ln\sqrt{e}$

$\ln\sqrt{e} = x$
$e^x = e^{1/2}$
$x = \frac{1}{2}$

c. $10^{\log 4}$

$\log 4 = x$
$10^x = 4$ so $10^{\log 4} = 4$
10 to a power and log are inverses

d. $\log 2 + \log 50$

$\log(2 \cdot 50) = \log 100$
2

e. $\log_4 192 - \log_4 3$

$\log_4\left(\frac{192}{3}\right)$
$\log_4 64 = 3$

f. $\ln\sqrt[5]{e^3}$

$\ln e^{3/5} = \frac{3}{5}\ln e = \frac{3}{5}$

2. Solve a. $\log_9\left(x^2 - x + 3\right) = \frac{1}{2}$

$x^2 - x + 3 = 9^{1/2}$
$x(x-1) = 0$
$x = 0, x = 1$

b. $\log_{36} x + \log_{36}(x-1) = \frac{1}{2}$

$\log_{36} x(x-1) = \frac{1}{2}$
$x(x-1) = 36^{1/2} = 6$
$x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$
Only $x = 3$ is in the domain

c. $\ln x - \ln(x-1) = 1$

$\ln\left(\frac{x}{x-1}\right) = 1$
$\frac{x}{x-1} = e \Rightarrow x = ex - e$
$x = \frac{e}{e-1}$

d. $5^x = 20$

$\log(5^x) = \log 20$
$x\log 5 = \log 20$
$x = \frac{\log 20}{\log 5}$ or $x = \frac{\ln 20}{\ln 5}$

e. $e^{-2x} = 5$

$\ln e^{-2x} = \ln 5$
$-2x = \ln 5 \Rightarrow x = \frac{-\ln 5}{2}$

f. $2^x = 3^{x-1}$

$\log(2^x) = \log(3^{x-1})$
$x\log 2 = (x-1)\log 3$
$x\log 2 = x\log 3 - \log 3 \Rightarrow x = \frac{\log 3}{\log 3 - \log 2}$

**Video Solving Log Equations ~**

https://www.youtube.com/watch?v=iBZ0ArS0ZVY&t=4s

**Video Properties of Exponents**

https://www.youtube.com/watch?v=7ryfCvDzIKY&t=117s

## J. Exponential and Logarithmic Functions – Assignment

*Simplify each. You should be able to do so without a calculator!*

88. $\frac{1}{4}$

89. $4$

90. $ln\ ln\ \frac{1}{\sqrt[3]{e^2}}$

91. $5^{40}$

92. $e^{ln12}$

93. $\frac{2}{3} + \frac{3}{32}$

94. $\frac{4}{3} - \ 12$

95. $\left(\sqrt{3}\right)^5$

*Solve. Be sure to check your solution in the original equation.*

96. $(3x - 8) = 2$

97. $(x^2 - x + 3) = \frac{1}{2}$

98. $log\ log\ (x - 3) + log\ log\ 5 = 2$

99. $(x + 3) - x\ = 2$

100. $3^{x-2} = 18$

101. $e^{3x+1} = 10$

## K. Radian and Degree Measures / Reference Angles / Reference Triangles

Use $\frac{180^\circ}{\pi\ radians}$ to convert to degrees.

**Example 1: Convert 300° to radians.**

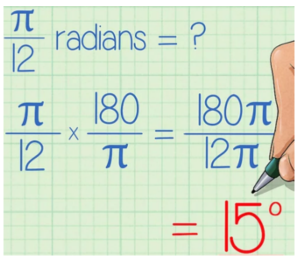
**Solution:**

*We know 180° = π radians = π^c or 1° = (π/180)^c*

*Hence, 300° = 300 × π/180 = 5π/3*

*Thus, 300° = 5π/3 radians*

*Example 2: Convert Radians to Degrees*




Use $\frac{\pi\ radians}{180^\circ}$ to convert to radians.

**Example 3: Convert –300° to radians.**

**Solution:**

*We know 180° = π radians = π^c or 1° = (π/180)^c*

*Hence, –300° = –300 × π/180 = –5π/3*

*Thus, –300° = –5π/3 radians*

*Example 4: Graph the angle in standard position. Draw the reference triangle. Determine the reference angle, then label the sides with the appropriate lengths.*

a.) sin225°

b.) $cos\frac{-7\pi}{6}$

Video Examples:

https://www.youtube.com/watch?v=A9AEzO9RnGo

## K. Radian and Degree Measure / Reference Angles / Reference Triangles – Assignment

*Convert to degrees:*

102. $\frac{5\pi}{6}$

103. $\frac{4\pi}{5}$

104. $\frac{-7\pi}{6}$

105. $\frac{2\pi}{3}$

106. $\frac{-3\pi}{2}$

107. 2.63 radians

*Convert to radians:*

108. 45°

109. – 17°

110. 237°

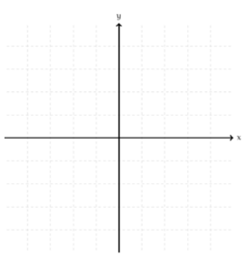
111. 180°

112. – 30°

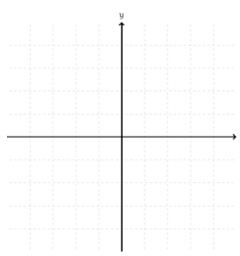
113. –225°

*Sketch the angle in Standard Position.*

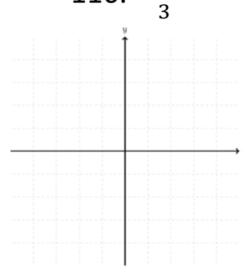
114. $\frac{11\pi}{6}$



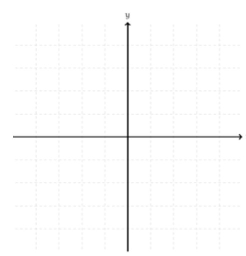
115. 230°



116. $\frac{-5\pi}{3}$

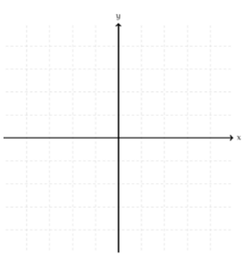


117. 1.8 rad

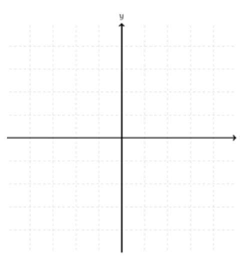


*Sketch the angle in standard position. Draw the reference triangle and label the sides.*

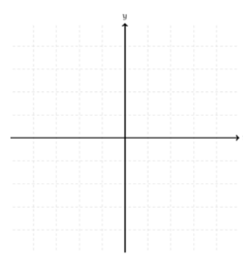
118. $\frac{2\pi}{3}$



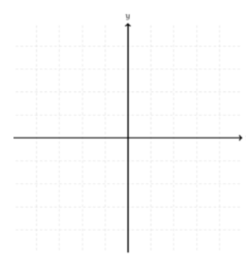
119. 225°



120. $\frac{-\pi}{4}$



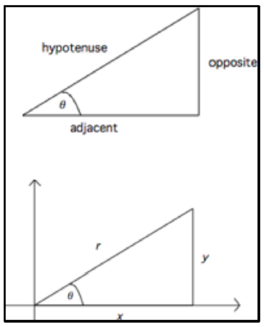
121. – 30°



## L. Right Triangle Trigonometry

Trigonometry is an integral part of AP calculus. Students must know the basic trig function definitions in terms of opposite, adjacent and hypotenuse as well as the definitions if the angle is in standard position.

Given a right triangle with one of the angles named $\theta$, and the sides of the triangle relative to $\theta$ named opposite ($y$), adjacent ($x$), and hypotenuse ($r$) we define the 6 trig functions to be:



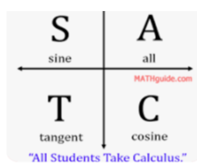

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hyptotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hyptotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

The Pythagorean theorem ties these variables together: $x^2 + y^2 = r^2$. Students should recognize right triangles with integer sides: 3-4-5, 5-12-13, 8-15-17, 7-24-25. Also any multiples of these sides are also sides of a right triangle. Since $r$ is the largest side of a right triangle, it can be shown that the range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$, the range of $\csc\theta$ and $\sec\theta$ is $(-\infty,-1]\cup[1,\infty)$ and the range of $\tan\theta$ and $\cot\theta$ is $(-\infty,\infty)$.

Also vital to master is the signs of the trig functions in the four quadrants. A good way to remember this is A – S – T – C where All trig functions are positive in the 1st quadrant, Sin is positive in the 2nd quadrant, Tan is positive in the 3rd quadrant and Cos is positive in the 4th quadrant.




1. Let be a point on the terminal side of $\theta$. Find the 6 trig functions of $\theta$. (Answers need not be rationalized).

a) $P(-8,6)$

$x=-8, y=6, r=10$

$\sin\theta = \frac{3}{5}$ $\quad \csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\quad \sec\theta = -\frac{5}{4}$

$\tan\theta = -\frac{3}{4}$ $\quad \cot\theta = -\frac{4}{3}$

b. $P(1,3)$

$x=1,\ y=3,\ r=\sqrt{10}$

$\sin\theta = \frac{3}{\sqrt{10}}$ $\quad \csc\theta = \frac{\sqrt{10}}{3}$

$\cos\theta = \frac{1}{\sqrt{10}}$ $\quad \sec\theta = \sqrt{10}$

$\tan\theta = 3$ $\quad \cot\theta = \frac{1}{3}$

c. $P\left(-\sqrt{10},-\sqrt{6}\right)$

$x=-\sqrt{10},\ y=-\sqrt{6},\ r=4$

$\sin\theta = -\frac{\sqrt{6}}{4}$ $\quad \csc\theta = -\frac{4}{\sqrt{6}}$

$\cos\theta = -\frac{\sqrt{10}}{4}$ $\quad \sec\theta = -\frac{4}{\sqrt{10}}$

$\tan\theta = \sqrt{\frac{3}{5}}$ $\quad \cot\theta = \sqrt{\frac{5}{3}}$

2. If $\cos\theta = \frac{2}{3}$, $\theta$ in quadrant IV, find $\sin\theta$ and $\tan\theta$

$x=2,\ r=3,\ y=-\sqrt{5}$

$\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = -\frac{\sqrt{5}}{2}$

3. If $\sec\theta = \sqrt{3}$ find $\sin\theta$ and $\tan\theta$

$\theta$ is in quadrant I or IV

$x=1,\ y=\pm\sqrt{2},\ r=\sqrt{3}$

$\sin\theta = \pm\sqrt{\frac{2}{3}},\ \tan\theta = \pm\sqrt{2}$

4. Is $3\cos\theta + 4 = 2$ possible?

$3\cos\theta = -2$

$\cos\theta = -\frac{2}{3}$ which is possible.

## L. Right Triangle Trigonometry – Assignment

Let *P* be a point on the terminal side of θ. Find the ratio of the 6 trig functions of θ. (You do not need to rationalize.)

122. *P*(15, 8)

123. *P*(– 2, 3)

124. $P\left(-2\sqrt{5}, -\sqrt{5}\right)$

125. If $tan\theta = \frac{12}{5}, and\ \theta$ is in quad. III, find sin θ and cos θ.

126. If $csc\theta = \frac{-6}{5}, and\ \theta$ is in quad. IV, Find cos θ and tan θ.

Determine which quadrant(s) where the following is true. Explain your reasoning.

127. $sin\theta > 0\ and\ cos\theta < 0$

128. $csc\theta < 0\ and\ cot\theta > 0$

129. All functions are negative.

## M. Special Angles

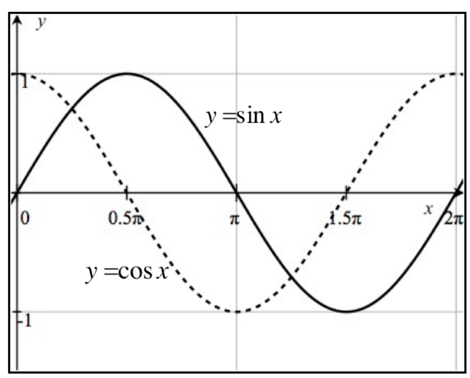
Students must be able to find trig functions of quadrant angles $(0,\ 90°, 180°, 270°)$ and special angles, those based on the $30°-60°-90°$ and $45°-45°-90°$ triangles.

First, for most calculus problems, angles are given and found in radians. Students must know how to convert degrees to radians and vice-versa. The relationship is $2\pi$ radians = $360°$ or $\pi$ radians = $180°$. Angles are assumed to be in radians so when an angle of $\frac{\pi}{3}$ is given, it is in radians. However, a student should be able to picture this angle as $\frac{180°}{3} = 60°$. It may be easier to think of angles in degrees than radians, but realize that unless specified, angle measurement must be written in radians. For instance, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

The trig functions of **quadrant angles** $\left(0,\ 90°, 180°, 270° \text{ or } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right)$ can quickly be found. Choose a point along the angle and realize that *r* is the distance from the origin to that point and always positive. Then use the definitions of the trig functions.

| $\theta$ | point | $x$ | $y$ | $r$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\csc\theta$ | $\sec\theta$ | $\cot\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $(1,0)$ | 1 | 0 | 1 | 0 | 1 | 0 | does not exist | 1 | does not exist |
| $\frac{\pi}{2}$ or $90°$ | $(0,1)$ | 0 | 1 | 1 | 1 | 0 | does not exist | 1 | does not exist | 0 |
| $\pi$ or $180°$ | $(-1,0)$ | -1 | 0 | 1 | 0 | -1 | 0 | does not exist | -1 | does not exist |
| $\frac{3\pi}{2}$ or $270°$ | $(0,-1)$ | 0 | -1 | 1 | -1 | 0 | Does not exist | -1 | does not exist | 0 |

If you picture the graphs of $y = \sin x$ and $y = \cos x$ as shown to the right, you need not memorize the table. You must know these graphs backwards and forwards.



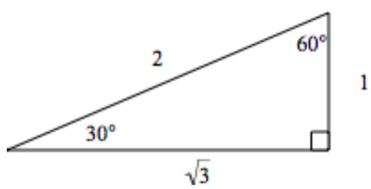

• Without looking at the table, find the value of

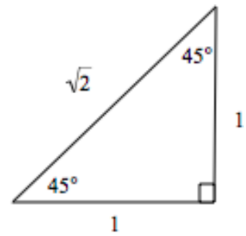
a. $5\cos 180° - 4\sin 270°$

$$\boxed{\begin{array}{l}5(-1)-4(-1)\\ -5+4=-1\end{array}}$$

b. $\left(\dfrac{8\sin\frac{\pi}{2} - 6\tan\pi}{5\sec\pi - \csc\frac{3\pi}{2}}\right)^2$ $\quad\boxed{\left[\dfrac{8(1)-6(0)}{5(-1)-(-1)}\right]^2 = \left(\dfrac{8}{-4}\right)^2 = 4}$

Because over half of the AP exam does not use a calculator, you must be able to determine trig functions of **special angles**. You must know the relationship of sides in both $30° - 60° - 90° \left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ and $45° - 45° - 90° \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ triangles.




In a $30° - 60° - 90° \left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ triangle, the ratio of sides is $1 - \sqrt{3} - 2$.

In a $45° - 45° - 90° \left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ triangle, the ratio of sides is $1 - 1 - \sqrt{2}$.

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $30° \left(\text{or } \frac{\pi}{6}\right)$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $45° \left(\text{or } \frac{\pi}{4}\right)$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $60° \left(\text{or } \frac{\pi}{3}\right)$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |

Special angles are any multiple of $30° \left(\frac{\pi}{6}\right)$ or $45° \left(\frac{\pi}{2}\right)$. To find trig functions of any of these angles, draw them and find the **reference angle** (the angle created with the $x$-axis). Although most problems in calculus will use radians, you might think easier using degrees. This will create one of the triangles above and trig functions can be found, remembering to include the sign based on the quadrant of the angle. Finally, if an angle is outside the range of $0°$ to $360° (0 \text{ to } 2\pi)$, you can always add or subtract $360° (2\pi)$ to find trig functions of that angle. These angles are called **co-terminal angles**. It should be pointed out that $390° \neq 30°$ but $\sin 390° = \sin 30°$.

• Find the exact value of the following

a. $4\sin 120° - 8\cos 570°$

Subtract $360°$ from $570°$
$4\sin 120° - 8\cos 210°$
$120°$ is in quadrant II with reference angle $60°$.
$210°$ is in quadrant III with reference angle $30°$.
$4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{-\sqrt{3}}{2}\right) = 6\sqrt{3}$

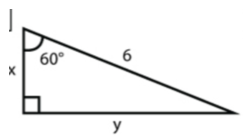
b. $\left(2\cos\pi - 5\tan\frac{7\pi}{4}\right)^2$

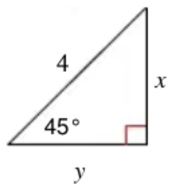
$(2\cos 180° - 5\tan 315°)^2$
$180°$ is a quadrant angle
$315°$ is in quadrant III with reference angle $45°$
$\left[2(-1) - 5(-1)\right]^2 = 9$

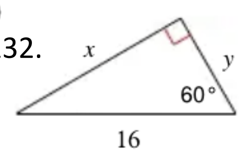
Video – Kooky Rapping Teacher Who Thinks He is Cool (solving special right triangles) https://www.youtube.com/watch?v=hftTj9RfuxM

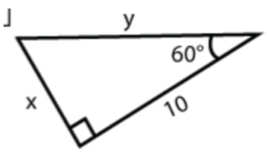
## M. Special Angles – Assignment

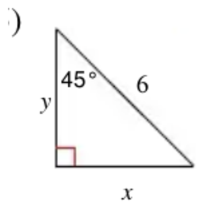
*Determine the value of x and y EXACTLY (no decimals – no calculator).*

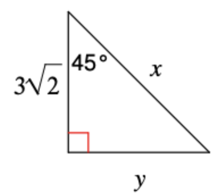
130. 


131. 


132. 


133. 


134. 


135. 


*Evaluate each WITHOUT using a calculator! (Draw reference triangles.)*

136. $sin^2 120° + cos^2 120°$

137. $2tan^2 300° + 3sin^2 150° - \ cos^2 180°$

138. $cot^2 135° - sin 210° + 5cos^2 225°$

139. $cot\ (-30°)\ + tan 600° - csc\ (-450°)$

140. $\left(cos\frac{2\pi}{3} - tan\frac{3\pi}{4}\right)^2$

141. Determine if the following is TRUE or FALSE.

$$sin\frac{\pi}{6} + sin\frac{\pi}{3} = sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

## N. Trigonometric Identities

Trig identities are equalities involving trig functions that are true for all values of the occurring angles. While you are not asked these identities specifically in calculus, knowing them can make some problems easier. The following chart gives the major trig identities that you should know. To prove trig identities, you usually start with the more involved expression and use algebraic rules and the fundamental trig identities. A good technique is to change all trig functions to sines and cosines.

Fundamental Trig Identities

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x$$

Sum Identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Double Angle Identities

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

• Verify the following identities.

1. $(\tan^2 x + 1)(\cos^2 - 1) = -\tan^2 x$

$$(\sec^2 x)(-\sin^2 x)$$
$$\left(\frac{1}{\cos^2 x}\right)(-\sin^2 x)$$
$$-\tan^2 x$$

2. $\sec x - \cos x = \sin x \tan x$

$$\frac{1}{\cos x} - \cos x\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$$
$$\sin x\left(\frac{\sin x}{\cos x}\right) = \sin x \tan x$$

3. $\dfrac{\cot^2 x}{1 + \csc x} = \dfrac{1 - \sin x}{\sin x}$

$$\left(\frac{\frac{\cos^2}{\sin^2 x}}{1 + \frac{1}{\sin x}}\right)\frac{\sin^2 x}{\sin^2 x} = \frac{\cos^2 x}{\sin^2 x + \sin x}$$
$$\frac{1 - \sin^2 x}{\sin x(1 + \sin x)} = \frac{(1 + \sin x)(1 - \sin x)}{\sin x(1 + \sin x)}$$
$$\frac{1 - \sin x}{\sin x}$$

4. $\dfrac{1 + \sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = 2\sec x$

$$\left(\frac{1 + \sin x}{\cos x}\right)\left(\frac{1 + \sin x}{1 + \sin x}\right) + \left(\frac{\cos x}{1 + \sin x}\right)\left(\frac{\cos x}{\cos x}\right)$$
$$\frac{1 + 2\sin x + \sin^2 + \cos^2 x}{\cos x(1 + \sin x)}$$
$$\frac{1 + 2\sin x + 1}{\cos x(1 + \sin x)} = \frac{2 + 2\sin x}{\cos x(1 + \sin x)}$$
$$\frac{2(1 + \sin x)}{\cos x(1 + \sin x)} = 2\sec x$$

**Verifying identities is really just a way to practice both memorizing the identities and the algebra needed for certain problems. In Calculus, you will need to be able to simplify expressions using identities. To simplify, try to write your final answer as one fraction, no complex fractions, one single term, and if possible, no fractions.

**Video Time!!**

https://www.youtube.com/watch?v=jkpRkOKkaCw

## N. Trigonometric Identities – Assignment

*Simplify using identities. Remember, try to write your answer as one term and no fractions if possible.*

142. $\frac{1}{cosx} =$

143. $\frac{1}{sec^2x} =$.

144. $\frac{1}{cotx} =$

145. $\frac{sinx}{cosx} =$

146. $cot^2x + 1 =$

147. $2sinxcosx =$

148. $cos^2x - 1 =$

149. $2cos^2x - 1 =$

150. $sin2B =$

151. $sec^2x - 1 =$

152. $\frac{cos^2x}{sin^2x} =$

153. $sinAcosB + cosAsinB =$

***Verify the following identities.***

154. $(1 + sinx)(1 - sinx) = cos^2x$

155. $sec^2x + 3 = tan^2x + 4$

156. $\frac{1-secx}{1-cosx} = -secx$

157. $\frac{1}{1+tanx} + \frac{1}{1+cotx} = 1$

158. $\frac{cosx-cosy}{sinx+siny} + \frac{sinx-siny}{cosx+cosy} = 0$

159. $\frac{sin^3x+cos^3x}{sinx+cosx} = 1 - sinxcosx$

## O. Solving Trig Equations

Trig equations are equations using trig functions. Typically they have many (or infinite) number of solutions so usually they are solved within a specific domain. Without calculators, answers are either quadrant angles or special angles, and again, they must be expressed in radians.

- Solve for $x$ on $[0,2\pi)$

1. $x\cos x = 3\cos x$

Do not divide by $\cos x$ as you will lose solutions

$$\cos x(x-3)=0$$

$$\cos x = 0 \qquad x-3=0$$

$$x=\frac{\pi}{2},\frac{3\pi}{2} \qquad x=3$$

You must work in radians.

Saying $x = 90°$ makes no sense.

2. $\tan x + \sin^2 x = 2 - \cos^2 x$

$$\tan x + \sin^2 x + \cos^2 x = 2$$

$$\tan x + 1 = 2$$

$$\tan x = 1$$

$$x=\frac{\pi}{4},\frac{5\pi}{4}$$

Two answers as tangent is positive in quadrants I and III.

3. $3\tan^2 x - 1 = 0$

$$3\tan^2 x = 1$$

$$\tan^2 x = \frac{1}{3}$$

$$\tan x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$

$$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$$

4. $3\cos x = 2\sin^2 x$

$$3\cos x = 2\left(1-\cos^2 x\right)$$

$$2\cos^2 x + 3\cos x - 2 = 0$$

$$(2\cos x - 1)(\cos x + 2) = 0$$

$$2\cos x = 1 \qquad \cos x = -2$$

$$\cos x = \frac{1}{2} \qquad \text{No solution}$$

$$x=\frac{\pi}{3},\frac{5\pi}{3}$$

More???

https://www.youtube.com/watch?v=kajoHHenS2Q

## O. Solving Trig Equations - Assignment

*Solve for x on [0, 2π)*

160. $sin^2x = sinx$

161. $4cos^2x - 1 = 0$

162. $sin^2x = 3cos^2x$

163. $2cos^2x - 1 =$

164. $sinx = cosx$

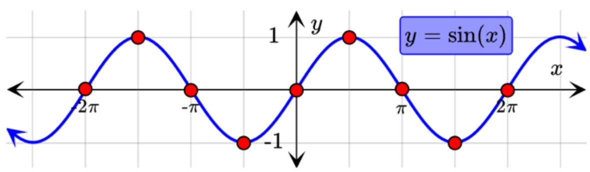
165. $2cos^2x + sinx - 1 = 0$

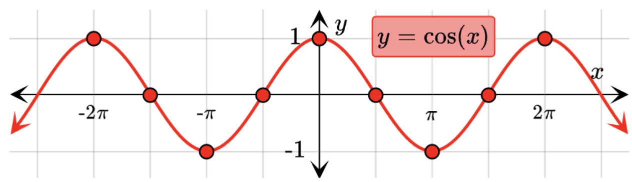
166. $3sinx = 2cos^2x$

## P. Graphing Trig Functions – Notes and Assignment

**Video:**

https://www.educreations.com/lesson/view/graphing-sine-and-cosine/62643682/




The graphs of $y = sinx\ and\ y = cosx$ have a period of 2π and an amplitude of 1. Use the parent graphs above to help you sketch a graph of each of the functions below. Recall:

$$y = Asin(Bx + C) + k$$

A = amplitude $\frac{2\pi}{B} = period$ $K$ = vertical shift

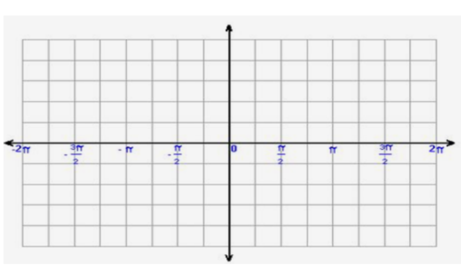
$\frac{C}{B} = phase\ shift$ (positive shifts left and negative shifts right)

*Determine the period, vertical shift, phase shift, and amplitude for each. Then graph two periods using your calculator.*

167. $f(x) = 5sinx$

Amp = ______ PS = ________

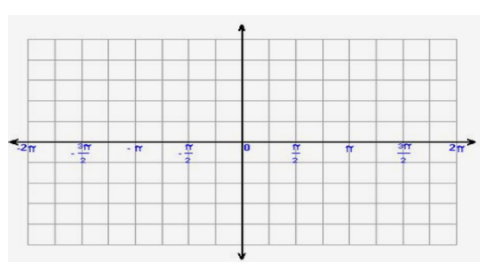
Period = _______ VS = _______



168. $f(x) = sin2x$

Amp = ______ PS = ________

Period = _______ VS = ________



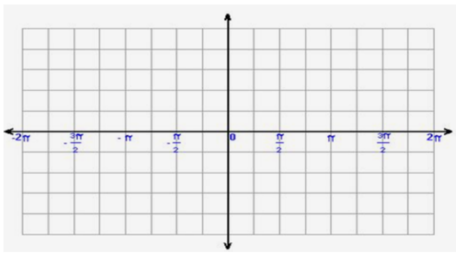
169. $f(x) = cos\ cos\ (x)\ - 3$

Amp = ______

PS = ________

Period = _______

VS = _______



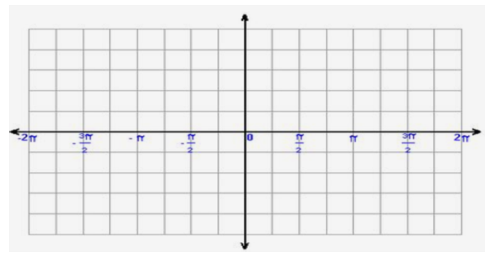
170. $f(x) = -cos\left(x + \frac{\pi}{4}\right)$

Amp = ______

PS = ________

Period = _______

VS = ________



## Q. Graphical Solutions

You have a shiny new calculator. So when are we going to use it? So far, no mention has been made of it. Yet, the calculator is a tool that is required in the AP calculus exam. For about 25% of the exam, a calculator is permitted. So it is vital you know how to use it.



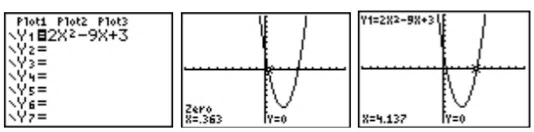
There are several settings on the calculator you should make. First, so you don't get into rounding difficulties, it is suggested that you set your calculator to three decimal places. That is a standard in AP calculus so it is best to get into the habit. To do so, press MODE and on the 2nd line, take it off FLOAT and change it to 3. And second, set your calculator to radian mode from the MODE screen. There may be times you might want to work in degrees but it is best to work in radians.

You must know how to graph functions. The best way to graph a function is to input the function using the Y= key. Set your XMIN and XMAX using the WINDOW key. Once you do that, you can see the graph's total behavior by pressing ZOOM 0. To evaluate a function at a specific value of $x$, the easiest way to do so is to press these keys: VARS → 1:Function 1 1:Y1 ( and input your $x$-value.
Other than basic calculations, and taking trig functions of angles, there are three calculator features you need to know: evaluating functions at values of $x$ and finding zeros of functions, which we know is finding where the function crosses the $x$-axis. The other is finding the point of intersection of two graphs. Both of these features are found on the TI-84+ calculator in the CALC menu 2ND TRACE. They are 1:value, 2: zero, and 5: intersect.

Solving equations using the calculator is accomplished by setting the equation equal to zero, graphing the function, and using the ZERO feature. To use it, press 2ND TRACE ZERO . You will be prompted to type in a number to the left of the approximate zero, to the right of the approximate zero, and a guess (for which you can press ENTER). You will then see the zero (the solution) on the screen.

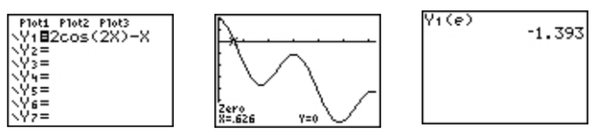
- Solve these equations graphically.

1. $2x^2 - 9x + 3 = 0$




This equation can be solved with the quadratic formula. $x = \dfrac{9 \pm \sqrt{81-24}}{4} = \dfrac{9 \pm \sqrt{57}}{4}$

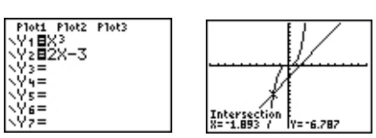
2. $2\cos 2x - x = 0$ on $[0, 2\pi)$ and find $2\cos(2e) - e$.



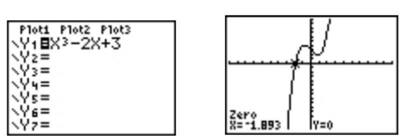

If this were the inequality $2\cos 2x - x > 0$, the answer would be $[0, 0.626)$.

3. Find the $x$-coordinate of the intersection of $y = x^3$ and $y = 2x - 3$

You can use the intersection feature.




Or set them equal to each other: $x^3 = 2x - 3$ or $x^3 - 2x + 3 = 0$



## Q. Graphical Solutions – Assignment

*Solve the following equations graphically (use your calculator!) Round any decimal answers to three (3) decimal places.*

171. $3x^3 - x - 5 = 0$

172. $3x^3 - 5x^2 + 4x - 1 = 0$

173. $2x^2 - 1 = 2$

174. $2\ ln\ ln\ (x + 1)\ = 5cosx\ \ on\ \ [0, 2\pi)$

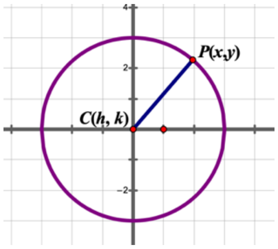
175. $x^4 - 9x^2 - 3x - 15 = 0$

176. $\frac{x^2-4x-4}{x^2+1} = 0\ \ on\ \ [0, 8]$

177. $xsinx^2\ = 0\ \ on\ [0, 3]$

178. $cos^{-1}x = x^2\ \ \ on\ \ [-1, 1]$

## R. Circles and Ellipses – Notes and Assignment




**Circles**

$$(x-h)^2 + (y-k)^2 = r^2$$

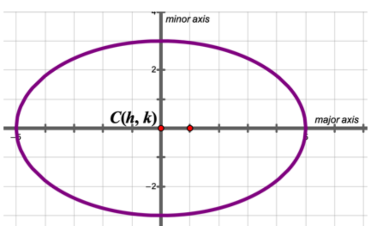
**Center (*h, k*)**

**r = radius**

For a circle centered at the origin, the equation is: $x^2 + y^2 = r^2$

Graphing Circles: https://www.youtube.com/watch?v=_VwhdOZMozE

**Ellipses**




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

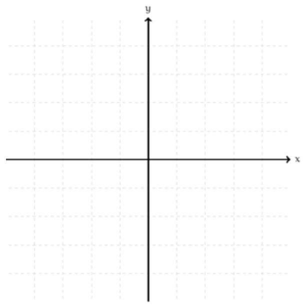
*a* = the distance from the center to the ellipse on the x-axis
*b* = the distance from the center to the ellipse on the *y*-axis.

For Calculus, we will not need to find the focus of the ellipse.

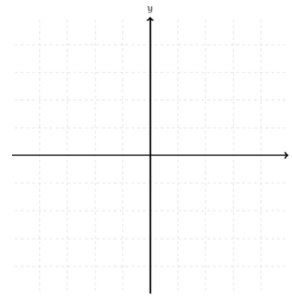
Graphing Ellipses: https://www.youtube.com/watch?v=euaw24U_r5o

*Graph the circles and ellipses below:*

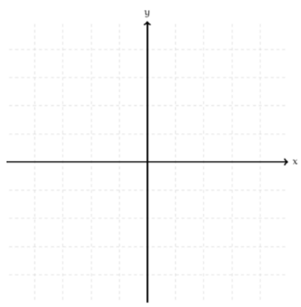
179. $x^2 + y^2 = 16$



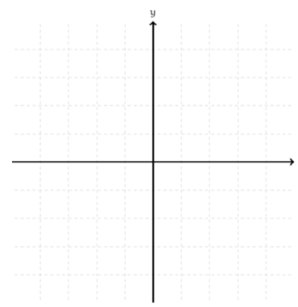

180. $x^2 + y^2 = 5$




181. $\frac{x^2}{1} + \frac{y^2}{9} = 1$



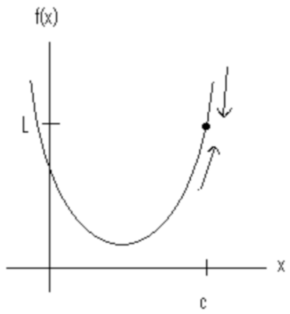

182. $\frac{x^2}{16} + \frac{y^2}{4} = 1$



## S. Limits

*Informal introduction to limits*:

$\lim_{x \to c} f(x) = L$ "the limit of a function *f(x)* as *x* approaches *c* is *L*"




As you approach c from the left or from the right, the value of the function becomes arbitrarily close to L.

*Limits can be determined by*:

- Examining the graph (graphically)
- Examining a table of values (numerically)
- Substituting the value of the function (analytically, we will explore this further in section 1.3)

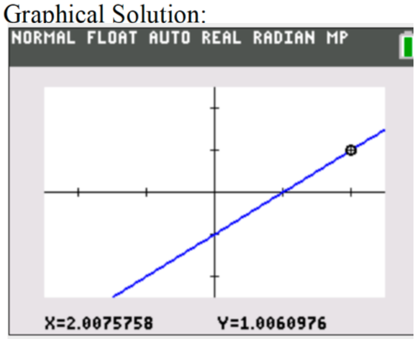
***Example 1:***

Analytical Solution:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

Factor: $\lim_{x \to 2} \frac{(x-2)(x-1)}{(x-2)}$
Cancel (x-2) then do direct substitution.

Graphical Solution:




Numerical Solution:

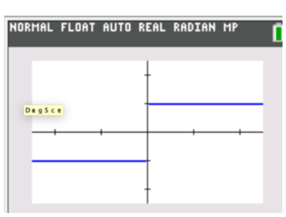
| x | y |
|---|---|
| 1.9 | .9 |
| 1.99 | .99 |
| 1.999 | .999 |
| 2 | Undefined |
| 2.001 | 1.001 |
| 2.01 | 1.01 |
| 2.1 | 1.1 |

### *Limits that do not exist*:

In some cases you will not be able to determine the limit of a function. The following are three examples of limits that do not exist.

***Case 1.*** The function f(x) may approach a different value from the right and from the left as x approaches *c*.

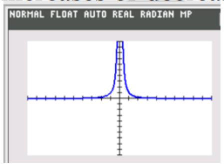
$$\lim_{x \to 0} \frac{|x|}{x}$$




Limit from left = -1 and limit from right = +1
Left and right limit do not agree so limit as x → 0 does not exist.

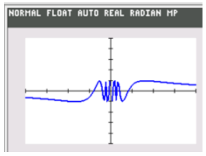
***Case 2.*** The function f(x) increases or decreases without bound as $x$ approaches $c$.

$$\lim_{x \to 0} \frac{1}{x^2}$$




***Case 3.*** The function f(x) oscillates between two values as $x$ approaches $c$.

$$\lim_{x \to 0} \left( \sin \frac{1}{x} \right)$$




***More Practice with Analytical Solutions of Limits:***

| *Limit:* | *General Properties of Limits:* |
| --- | --- |
| 1. $\lim_{x \to 3} 5 =$ | $\lim_{x \to c} b = b$ |

(Hint: Use Direct Substitution whenever possible. You can direct substitute with any polynomial function or rational function with a non-zero denominator.)

| | |
| --- | --- |
| 2. $\lim_{x \to 2} 3x^2 =$ | $\lim_{x \to c} [b \cdot f(x)] = b \lim_{x \to c} [f(x)]$ |
| 3. $\lim_{x \to 0} \left[ (5x+1) + x^2 \right] =$ | $\lim_{x \to c} \left[ f(x) \pm g(x) \right] = \lim_{x \to c} \left[ f(x) \right] + \lim_{x \to c} \left[ g(x) \right]$ |
| 4. $\lim_{x \to \pi} [(\cos x)(\sin x)] =$ | $\lim_{x \to c} \left[ f(x) \cdot g(x) \right] = \lim_{x \to c} \left[ f(x) \right] \bullet \lim_{x \to c} \left[ \cdot g(x) \right]$ |
| 5. $\lim_{x \to 5} \frac{x^2 - 2x - 5}{x+1}$ | $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ |

(Hint: Factor and simplify before applying the limit.)

5. $\lim_{x \to 4} \frac{2x^2 - 5x - 12}{x - 4}$

6. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(Hint: Rationalize the numerator or denominator as needed, then take the limit.)

7. $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

## *Special Trig Limits:*

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0 \qquad \lim_{x\to 0}\frac{\sin x}{x}=1$$

**Definition of Continuity:**

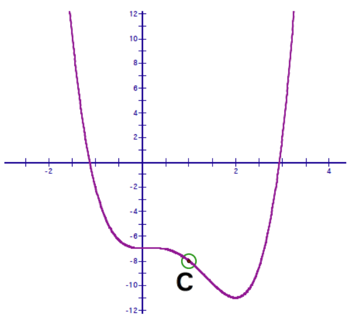
Informal: A function is continuous if its graph can be drawn without lifting the pencil from the paper.

Formal Definition of Continuity: A function is continuous at *c* if:

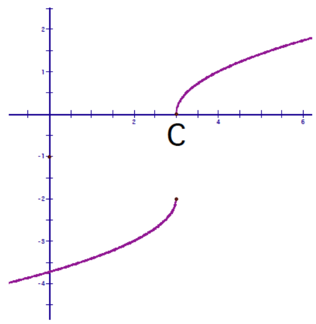
1) *f(c)* is defined
2) $\lim_{x\to c} f(x)$ exists
3) $\lim_{x\to c} f(x) = f(c)$

**Function is NOT continues at c:**

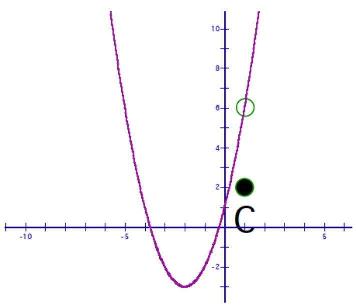
**Condition 1 is not met**




**Condition 2 is not met**




**Condition 3 is not met**




**HOLE = Removable discontinuity** **JUMP or VERTICAL ASYMPTOTE = non-removable**

**One Sided Limits:**

$\lim_{x\to c^+} f(x) = L$ limit as x approaches c from the right is L

$\lim_{x\to c^-} f(x) = L$ limit as x approaches c from the left is L

**Existence of a Limit**

Let *f* be a function and let *c* and *L* be real numbers. The limit of *f(x)* as *x* approaches *c* is *L* if and only if,

$\lim_{x\to c^-} f(x) = L$ **and** $\lim_{x\to c^+} f(x) = L$

**Video for Notes:**

https://www.educreations.com/lesson/view/limits/62643492/

**One-Sided Limits and Continuity**

https://www.youtube.com/watch?v=fc0lxNSgawI&t=20s

## S. Limits – Assignment

*Determine each limit numerically.*

183. $\lim_{x \to 4} \left( \frac{x-4}{x^2-3x-4} \right)$

| *x* | 3.9 | 3.99 | 3.999 | 4.001 | 4.01 | 4.1 |
|---|---|---|---|---|---|---|
| *f(x)* | | | | | | |

184. $\lim_{x \to -5} \left( \frac{\sqrt{4-x}-3}{x+5} \right)$

| *x* | -5.1 | -5.01 | -5.001 | -4.999 | -4.99 | -4.9 |
|---|---|---|---|---|---|---|
| *f(x)* | | | | | | |

*Find each limit graphically. Use your graphing calculator to assist in graphing.*

185. $\lim_{x \to 0} (\cos x) =$

186. $\lim_{x \to 5} \left( \frac{2}{x-5} \right) =$

187. $\lim_{x \to 1} f(x) =$

$$f(x) = \begin{cases} x^2 + 3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

*Evaluate each limit analytically. Use direct substitution when possible. If needed, rearrange the expression so that you can do direct substitution.*

188. $\lim_{x \to 2} (4x^2 + 3)$

189. $\lim_{x \to 1} \frac{x^2+x+2}{x+1}$

190. $\lim_{x \to 0} \sqrt{x^2 + 4}$

191. $\lim_{x \to 1} \left( \frac{x^2-1}{x-1} \right)$ (HINT: Factor and simplify)

192. $\lim_{x \to -3} \left( \frac{x^2+x-6}{x+3} \right)$

193. $\lim_{x \to \pi}(cos x)$

194. $\lim_{x \to 0}\left(\frac{\sqrt{x+1}-1}{x}\right)$ (HINT: Rationalize the numerator)

195. $\lim_{x \to 1}\left(\frac{3-x}{x^2-9}\right)$

196. $\lim_{h \to 0}\left(\frac{2(x+h)-2x}{h}\right)$

*One Sided Limits – Find the limit if it exists. First, try to solve for the overall limit. If an overall limit exists, then the one-sided limit will be the same as the overall limit. If not, use the graph and/or a table of values to evaluate one-sided limits.*

197. $\lim_{x \to 5^+}\left(\frac{x-5}{x^2-25}\right)$

198. $\lim_{x \to -3^-}\left(\frac{x}{\sqrt{x^2-9}}\right)$

199. $\lim_{x \to 10^+}\frac{|x-10|}{x-10}$

200. $\lim_{x \to 5^-}\left(\frac{-3}{x+5}\right)$